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DEDUCTION OF GENERALISED KRICHHOFF'S LAWS FROM THE BASIC PRINCIPLES OF ELECTROMAGNETISM

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ABSTRACT The existence of solution of the classical boundary value problem of steady current flow in a continuous semi-conducting medium under Ohm's Law is proved from the Fredholm Theory of Integral Equations Though the solution of the integral equations remains arbitrary, the solution of the boundary value problem has been shown to be unique as it should be from prima facie arguments A unique relation between the currents through the electrodes and their potentials is also established Further, an expression for the cross-currents flowing between two electrodes is deduced from theory leading to the usual Kirchhoff's Laws for line conductors

INTRODUCTION

The well-known Krichhoff's Laws describe the current voltage relationships in a net-work containing resistors These laws are based on (a) the macroscopic (Jeans, 1923) Ohm's Law which says that the total current flowing across a resistor is proportional to the difference of potential at its ends, and secondly (b) on the principle of steady state condition of electromagnetism under which any free volume charge does not appear The two laws of Kirchhoff are obvious consequences of these two principles The question of deduction of these two accepted 'laws' of Kirchhoff appears to be somewhat preposterous at first sight But there are two things which are normally overlooked, viz., the resistors in Kirchhoff's Laws are 'wires' which are idealised one-dimensional lines and Ohm's Law which is applied to these wires" is but a limiting approximation of the microscopic Ohm's Law for continuous media Strictly speaking, the Laws are to be deduced from the fundamental electromagnetic equations for a continuous resistive medium under the steady state conditions and then extended to the 'line' conductors as a limiting case. As this was not done before, the natural form of the Kirchhoff's Laws never became apparent As a result in the accepted form of Kirchhoff's Laws there are numerous redundant variables, which are difficult to eliminate in actual calculations. The powerful topological tool of Graph Theory has in recent years been extensively employed by circuit analysts to remove this difficulty We shall show in this paper that the Krichhoff's Laws take a simple natural form if deduced as a particular case of the more general laws for the continuous medium, which we have

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established from the fundamental equations of electromagnetism. The laws take a neater and convenient from which all the redundant variables disappear. The accepted form of the Kirchhoff's Laws can at once be deduced from these general relationships as a special case, but the reverse process is not at all obvious, because of the presence of the redundant variables.

The problem of steady current flow in a semi-conducting medium obeying Ohm's Law was first formulated by Kriehhoff (1893) He showed from physical principles that the currents through the electrodes and their potentials are connected through linear relations, as in the case of charges and potentials for a system of conductors shown earlier by Maxwell The main purpose of this work is to show rigorously that the solution of the boundary value problem of Kirchhoff always exists and though the fundamental solution has some irremovable arbitrainess, there exists a unique linear relation between the currents and potentials on the electrodes The existence of cross-currents between any two electrodes and their relation with the potentials are also deduced as a consequence from the existence theorem.

STATEMENT OF THE PROBLEM

In a finite semi-conducting medium of uniform specific conductivity κ let there be m electrodes The electrodes are perfect conductors by definition. Let

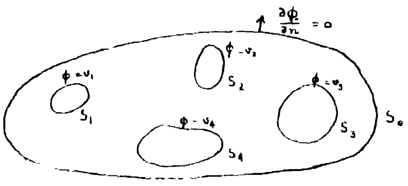


Fig. 1

the bounding surfaces of the electrodes be $S_1, S_2 \dots S_m$ Let S_0 be the bounding surface of the medium. The surfaces $S_1, S_2, \dots S_m$ are all enclosed within the outer surface S_0 Let $v_1, v_2, v_3, \dots v_m$ be the constant potentials on the respective electrode surfaces. Let $\vec{c}$ be the current density at any point P of the medium and $\phi(P)$ be the potential at P The steady state condition gives,

$$\operatorname{div} \vec{c} = 0 \quad \dots (1)$$

and Ohm's Law gives,

$$\vec{c} = -\kappa \operatorname{grad} \phi \quad \dots (2)$$

From these equations, follows the Laplace's Equation

$$\nabla^2\phi = 0 \quad \dots (3)$$

The boundary conditions for the problem are

$$\phi = \text{constant} = v_1, v_2, \ldots v_m \qquad \ldots \quad (4.1)$$

on the electrode surfaces $S_1, S_2, \ldots S_m$ and the normal component of the current density e_n on the enclosing surface S_0 shall vanish because no current can go out of the medium that is

$$e_n = -\kappa \left(\frac{\partial \phi}{\partial n}\right)_{S_0} = 0 \quad \text{on } S_0 \qquad \ldots \quad (4.2)$$

If a solution of this boundary values problem exists then it can be shown by Green's Theorem that this must be unique (see Appendix I). Further it must satisfy the equation of constraint, viz

$$\sum_{i=1}^{m} \iint_{S_i} \left(\frac{\partial \phi}{\partial n}\right)_{S_i} dS = \iiint \nabla^2 \phi \, d\tau = 0 \qquad (5)$$

PROOF OF THE EXISTENCE OF SOLUTION

Following the well-known method of Fredholm for the proof of existence of solution of the Dirichlet's and Neumann's problems in Potential Theory, we shall show that the solution of the boundary value problem exists. In the following we shall follow the notation of Sternberg (1952). Let us assume that the potential is due to a surface distribution $\sigma(Q)$ on the surfaces where $\sigma_0(Q)$ shall indicate the surface density on the outer enclosing surface S_0, $\sigma_1(Q)$ be the same on the electrode surface S_1 and so on. Obviously, $\sigma(Q)$ can be looked upon as an $(m+1)$ dimensional vector having the $(m+1)$ components $\sigma_0(Q), \sigma_1(Q), \ldots \sigma_m(Q)$ on the surfaces and has values only if Q is a point on the surfaces. If such a surface distribution exists in general, which satisfies the boundary conditions then the solution of the problem exists also. Further, such a solution has to be unique.

The potential at any point P inside the region is given by,

$$\phi(P) = \sum_{i=0}^{m} \iint_{S_i} \frac{\sigma_i(Q)}{r_P} \, dS \qquad . \quad (6)$$

This potential is continuous in crossing the surfaces $S_1, S_2, \ldots S_m$. Further by a well-known theorem of the Newtonian Potential Theory, the discontinuity of the normal derivative of the potential is given by (Stern berg et al. 1952).

$$\frac{\partial \phi_-}{\partial n} = \frac{\partial \phi}{\partial n_s} + 2\pi\sigma(s) \qquad \ldots \quad (7.1)$$

$$\frac{\partial \phi_+}{\partial n} = \frac{\partial \phi}{\partial n_s} - 2\pi\sigma(s) \qquad \ldots \quad (7.2)$$

where $\frac{\partial \phi_-}{\partial n}$ means the normal derivative from inside a closed surface, $\frac{\partial \phi_+}{\partial n}$ meaning from outside the surface, and $\frac{\partial \phi}{\partial n_s}$ means the value of the normal derivative on the surface. Thus, for any surface S_j

$$\left(\frac{\partial \phi_-}{\partial n}\right)_{S_j} = 2\pi\sigma_j(s) + \sum_{i=0}^{m} \frac{\partial}{\partial n_s} \int\int \frac{\sigma_i(Q)}{r_{sQ}}\, dS$$

Now, by the boundary conditions (4 2),

$$0 = \left(\frac{\partial \phi_-}{\partial n}\right)_{S_0} = 2\pi\sigma_0(s) + \sum_{i=0}^{m} \frac{\partial}{\partial n_s} \int\int_{S_i} \frac{\sigma_i(Q)}{r_{sQ}}\, dS$$

Further, since the potential function $\phi(P)$ is constant on the surfaces $S_1, S_2, \ldots S_m$ by the boundary condition (4.1), then it must be constant throughout the volume enclosed by each of these closed surfaces according to the continuity property of the potential function for a surfaces distribution Therefore,

$$\left(\frac{\partial \phi_-}{\partial n}\right)_{S_i} = 0, \quad i = 0, 1, 2, \ldots m \qquad \ldots \quad (8)$$

for each surface.

Thus by (7 1) $\sigma_j(s)$ must satisfy the same homogeneous integral equation on all the surfaces,

$$2\pi\sigma_j(s) + \sum_{i=0}^{m} \frac{\partial}{\partial n_s} \int\int_{S_i} \frac{\sigma_i(Q)}{r_{sQ}}\, dS = 0 \qquad \ldots \quad (9)$$

If non-trivial solution of this system of homogeneous integral equations exists, then the potential formed from this surface distribution will automatically satisfy the boundary conditions (4 1) and (4.2), as also the equation of constraint (5). For integrating the surface density $\sigma_0(Q)$ over the enclosing surface S_0 in (9) gives

$$2\pi \int\int_{S_0} \sigma_0(s) dS + \sum_{i=0}^{m} \int\int_{S_i} \sigma_i(Q) dS \int\int_{S_0} \frac{\partial}{\partial n_s} \left(\frac{1}{r_{sQ}}\right) dS =$$

Now, by Gauss Theorem, $\frac{1}{2\pi} \int\int_{S_0} \frac{\partial}{\partial n_s} \left(\frac{1}{r_{sQ}}\right) dS = -1$ when Q is an S_0

$= -2$ " " " " S_i

Thus,

$$\sum_{i=1}^{m} \int\int_{S_i} \sigma_i(Q) dS = 0$$

But,

$$\sigma_i(Q) = -\frac{1}{4\pi}\left(\frac{\partial\phi_i}{\partial n}\right)_{S_i}$$

$$i = 1, 2, 3, \ldots m$$

Or,

$$\sum_{i=1}^{m}\int\int_{S_i}\left(\frac{\partial\phi_i}{\partial n}\right)_{S_i} dS = 0$$

The problem is thus nearly analogous to the Neumann's Interior Problem of Potential Theory. Following the steps of Fredholm we shall now show that the solution of these coupled system of $(m+1)$ homogeneous integral equations exists in general and there are exactly $(m+1))$ linearly independent solutions corresponding to the eigen value -1.

Following Sternberg (1952) let us denote by

$$K(Q, s) = \frac{1}{2\pi}\frac{\partial}{\partial n_s}\left(\frac{1}{r_{sQ}}\right) \quad . \quad (10)$$

$$K(s, Q) = \frac{1}{2\pi}\frac{\partial}{\partial n_Q}\left(\frac{1}{r_{sQ}}\right) \quad . \quad (10)$$

then the equation (9) takes the form

$$\sigma_j(s) + \sum_{i=0}^{m}\int\int_{S_i} K(Q, s)\sigma_i(Q)dS = 0 \quad . \quad (11)$$

The corresponding transposed system of integral equations is,

$$\mu_j(s) + \sum_{i=0}^{m}\int\int_{S_i} K(s, Q)\mu_i(Q)dS = 0 \quad \ldots \quad (12)$$

By Gauss's Theorem,

$\mu_j(s) = 1$ and $\mu_i(s) = 0$ is a non-trivial solution. Thus,
$i \neq j$

$$\vec{\mu}(s) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \\ \\ 1 \end{bmatrix} \quad . \quad (13)$$

are $(m+1)$ linearly independent solutions corresponding to the eigen value-1. Therefore by Fredholm's theorem $(m+1)$ linearly independent solutions of the transposed homogenous integral equation (11) always exist. Thus $(m+1)$ linearly independent solutions of the fundamental integral equation (9) exist

General solution will be a linear combination of these $(m+1)$ linearly independent solutions, with $(m+1)$ arbitrary constants to be determined from the values of the potential given on the m electrode surfaces. One arbitrary constant cannot be determined because the boundary condition on the enclosing surface (0th surface) is satisfied automatically through the integral equations (9). But by the Uniqueness Theorem the potential function should be independent of this undeterminable constant

Further each fundamental solution of the integral equation (11) must satisfy the equation of constraint (5) automatically because expression (6) satisfies Laplace's Equation

Let
$$\overline{\nu_0}(s),\ \overline{\nu_1}(s)\ \overline{\nu_2}(s),\ \ldots,\ \overline{\nu_m}(s) \qquad . \quad (14)$$

be the $(m+1)$ linearly independent vector solutions of the integral equation (11) Each of these solutions satisfies the equation of constraint (5) Let us call by $N(s)$ the $(m+1)\times(m+1)$ matrix,

$$N(s) = \left(\overrightarrow{\nu_0}(s)\ \overrightarrow{\nu_1}(s),\ldots,\ \overrightarrow{\nu_m}(s)\right) \qquad .. \quad (15)$$

The surface density $\overline{\sigma}(s)$ will be a linear combination of these solutions, viz.

$$\begin{aligned}\overline{\sigma}(s) &= x_0\overline{\nu_0}(s)+x_1\overline{\nu_1}(s)+\ \ldots\ +x_m\overline{\nu_m}(s)\\ &= N(s)\overrightarrow{x}\end{aligned} \qquad .. \quad (16)$$

where $x_0, x_1, x_2, \ldots x_m$ are $(m+1)$ arbitrary constants to be determined from the boundary conditions of the problem The potential at any point P is thus

$$\begin{aligned}\phi(P) &= \left[\iint \left(\frac{1}{r_{pq_0}},\ \frac{1}{r_{pq_1}}\ \ldots\ \frac{1}{r_{pq_m}}\right) N(Q)dS\right]\overrightarrow{x}\\ &= \overrightarrow{\rho'}(P)\overrightarrow{x} \quad \text{say,}\end{aligned} \qquad .. \quad (17)$$

where $\left(\frac{1}{r_{pq_0}},\ \frac{1}{r_{pq_1}}\ \ldots\ \frac{1}{r_{pq_m}}\right)$ is a row vector and the row vector

$$\overrightarrow{\rho'}(P) = \iint \left(\frac{1}{r_{PQ_0}},\ \frac{1}{r_{PQ_1}},\ \ldots\ \frac{1}{r_{PQ_m}}\right) N(Q)dS \qquad .. \quad (18)$$

is the fundamental solution of the problem from which particular solutions are formed with different values of $\vec{x}$. According to the boundary conditions (4.1) $\phi(P)$ assumes the values $v_1, v_2, \ldots v_m$ on the electrodes. This can be expressed vectorically as

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} \int \int R(Q)\, N(Q) dS \end{bmatrix} \vec{x} \qquad \ldots \quad (19)$$

where $R(Q)$ is a rectangular matrix of m rows and $(m+1)$ columns, viz.,

$$R(Q) = \begin{bmatrix} \frac{1}{rs_1Q_0}, & \frac{1}{rs_1Q_1}, & \ldots & \frac{1}{rs_1Q_m} \\ \frac{1}{rs_2Q_0} & \frac{1}{rs_2Q_1}, & \ldots & \frac{1}{rs_2Q_m} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{1}{rs_mQ_0}, & \frac{1}{rs_mQ_1}, & \ldots & \frac{1}{rs_mQ_m} \end{bmatrix} \qquad \ldots \quad (20)$$

Let us denote the matrix with constant coefficients

$$\int \int R(Q)\, N(Q) dS = \underset{m\times(m+1)}{P} \qquad (21)$$

for brevity. Here P is obviously a rectangular matrix of m rows and $(m+1)$ columns. The coefficients of this matrix are completely determined from the fundamental solutions of the integral equation and from the geometry of the system. The $(m+1)$ arbitray constants are to be determined from the m equations (19) viz.,

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = P \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_m \end{bmatrix} = P\vec{x} \qquad \ldots \quad (22)$$

It is obvious that only m of the $(m+1)$ arbitrary constants $(x_0, x_1, x_2, \ldots, x_m)$ can be determined from these m equations in terms of $v_1, v_2, v_3 \ldots v_m$. One abritrary constant will remain undetermined The solution therefore remains apparently arbitrary for one arbitrary constant, which it should not according to the uniqueness theorem.

We shall show later, that though the surface densities cannot be uniquely determined, the matrix P satisfies some general conditions so that the potential function becomes unique Before doing this, we shall show that a unique relation between the total currents through the electrodes and their potentials exists, if we make use of the Green's Reciprocal Theorem. The total current i_l on the l-th electrode is given by,

$$i_l = -\frac{\kappa}{4\pi}\iint\limits_{S_l}\left(\frac{\partial\phi}{\partial n}\right) dS = \kappa \iint\limits_{S_l} \sigma_l(Q)dS$$

$$= \kappa \left(\iint N(Q)d\overrightarrow{S}x \right)_l \qquad \ldots \quad (23)$$

Let us call the $m \times (m+1)$ matrix

$$\iint N^0(Q)dS = Q \qquad \ldots \quad (24)$$

$N^0(Q)$ being formed from $N(Q)$ by removing its first row.

Thus, $$\overrightarrow{i} = \kappa Q\overrightarrow{x} \qquad \ldots \quad (25)$$

By the equation of constraint (5) the sum of the elements of a column of must be zero, that is, the rows of Q are linearly dependent.

That is

$$(1, 1, \ldots 1)Q = 0 \qquad \ldots \quad (26)$$

or $$Q'\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = 0 \qquad \ldots \quad (26')$$

UNIQUE RELATION BETWEEN CURRENTS AND POTENTIALS

Let $\overrightarrow{w}$ be another system of potentials and $\overrightarrow{j}$ the corresponding currents and $\psi(P)$ be the corresponding potential at a point P.

Then by Green's Theorem,

$$0 = \iiint (\psi\nabla^2\phi - \phi\nabla^2\psi)d\tau = \sum_{l=1}^{m}\iint\limits_{S_l}\left(\chi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right) dS$$

$$= -\frac{4\pi}{\kappa}\sum_{l=1}^{m}(w_l i_l - v_l j_l)$$

Therefore

$$\sum_i w_i i_i = \sum_i v_i j_i$$

or $$\vec{w}'\,\vec{i} = \vec{i}'\,\vec{w} = \vec{v}'\,\vec{j} = \vec{j}'\,\vec{v}$$

Now $$\vec{v} = P\vec{x} \quad \text{and} \quad \vec{w} = P\vec{y} \quad \text{(say)}$$

$$\vec{i} = \kappa Q\vec{x} \quad \text{and} \quad \vec{j} = \kappa Q\vec{y} \text{ (say)}$$

Thus, $$\vec{w}'\,\vec{i} = \kappa\,\vec{y}'P'Q\vec{x} \quad \vec{j}'\,\vec{v} = \kappa\vec{y}'Q'P\vec{x}$$

or $$\vec{y}'P'Q\vec{x} = \vec{y}'Q'P\vec{x}$$

for any arbitrary $\vec{x}$ and $\vec{y}$. Therefore

$$P'Q = Q'P = (P'Q)' \tag{27}$$

or $P'Q$ is a symmetric matrix of $(m+1)\times(m+1)$ order.
Again from (22) (25) and (27)

$$Q'\vec{v} = Q'P\vec{x} = P'Q\vec{x} = \frac{1}{\kappa}P'\vec{i}$$

Thus we obtain the important result.

$$\kappa Q'\vec{v} = P'\vec{i} \tag{28}$$

That is, the relation between the potentials on the electrodes and the currents flowing through them is unique and free from the inherent arbitrariness.

The currents $\vec{i}$ can be linearly expressed in terms of the potentials $\vec{v}$ from this equation (28). Multiplying both sides of equation (28) by P and denoting by U the Legendre Transform of P i.e.,

$$U = P\,P' \qquad \ldots \quad (29)$$

we obtain

$$\kappa PQ'\vec{v} = P\,P'\vec{i}$$

The $(m \times m)$ matrix U is non-singular (See Appendix II), and thus

$$\vec{i} = \kappa U^{-1}PQ'\vec{v} \qquad \ldots \quad (30)$$

It can also be shown that the matrix $U^{-1}PQ'$ is symmetric (Appendix II) Thus,

$$\vec{i} = \kappa U^{-1}PQ'\vec{v} = \kappa QP'U^{-1}\vec{v} \quad \ldots \quad (31)$$

When the potentials $v_1, v_2 \ldots v_m$ on the electrodes have the same value, say v_0 then it follows from (26′) that

$$\vec{i} = \kappa U^{-1}PQ' \begin{bmatrix} 1 \\ 1 \\ \\ 1 \end{bmatrix} \vec{v_0} = 0$$

which means physically that no current will flow through the electrodes, if they are kept at the same potential

COMPLETE SOLUTION OF THE PROBLEM

It has been shown in Section 3, that all the $(m+1)$ arbitrary constants x_0, $x_1, x_2 \ldots, x_m$ cannot be determined from the boundary conditions and one of these will remain arbitrary. Instead of choosing any particular one of these constants $x_0, x_1 \ldots x_m$ as arbitrary we can introduce a new arbitrary constant which appears in a natural way. There are m equations in (22), and there are $(m+1)$ unknowns to be determined Multiplying both sides of equation (22) by P' which is a $(m+1) \times m$ matrix, we get $(m+1)$ linear equations .

$$\overrightarrow{P'v} = \overrightarrow{P'Px}$$

Here $P'P$ is a $(m+1) \times (m+1)$ matrix and is of rank m. It can be shown that the solution of these equations always exists (See Appendix III) and the solution is,

$$\vec{x} = P'U^{-1}\vec{v} + \alpha\vec{g} \quad \ldots \quad (32)$$

where α is an arbitrary constant and the vector $\vec{g}$ having $(m+1)$ elements satisfies the homogeneous equation

$$P'P\vec{g} = 0 \quad \ldots \quad (33)$$

Further $\vec{g}$ is not a null-vector

This vector $\vec{g}$ has some interesting properties Let us form a particular surface distribution $\vec{\omega}(s)$ by putting $\vec{x} = \vec{g}$ in (16),

$$\vec{\omega}(s) = N(s)\vec{g} \quad \ldots \quad (34)$$

Since (See Appendix III)

$$\overrightarrow{Pg} = 0, \quad \overrightarrow{Qg} = 0 \quad \ldots \quad (35)$$

$$\vec{v} = \overrightarrow{Pg} = 0 \quad \text{and} \quad \vec{i} = \kappa\overrightarrow{Qg} = 0$$

This means that this particular non-trivial surface-distribution $\vec{\omega}(s)$ gives rise to null potentials on the electrode surfaces $S_1, S_2 \ldots S_m$ at the same time the currents flowing through the electrodes also vanish. This appears to be paradoxial at first sight. But we can identify this peculiar distribution with the help of relations (7.1) and (7.2). Since the potential $\phi(s)$ vanishes on the electrode surfaces so by Green's Theorem,

$$\iiint(|\operatorname{grad}\phi|^2 + \phi\nabla^2\phi)d\tau = \iiint \operatorname{div}\,(\phi\operatorname{grad}\phi)d\tau$$

$$= \sum_{i=0}^{m} \iint_{S_i} \phi\,\frac{\partial\phi}{\partial n}\,dS$$

$$= 0$$

Thus, $|\operatorname{grad}\phi|^2 = 0$ everywhere which means that $\phi = 0$ everywhere inside S_0. This means particularly,

$$\left(\frac{\partial\phi_+}{\partial n}\right)_{S_i} = 0$$

From (7.1) and (7.2) it follows,

$$\omega_i(s) = 0, \quad i = 1, 2, 3, \ldots m \tag{36}$$

and from (11),

$$\omega_0(s) + \iint_{S_0} K(Q, s)\,\omega_0(Q)dS = 0 \tag{37}$$

This integral equation has a non-trivial solution according to the Fredholm Theory.* $\omega_0(s)$ therefore exists mathematically. Since the potential function vanishes throughout the region inside S_0 and as it is continuous on S_0, the enclosing surface S_0 becomes an equipotential surface with zero potential due to this particular distribution. The surface distribution $\omega_0(s)$ is identical to the distribution which will be formed if the surface S_0 were a perfect conductor, grounded and subject to inductive influences of the field due to outside bodies or in other words if S_0 were an electrical shield for the internal conductors. Thus such a surface distri-bution physically can exist. Since this surface distribution $\vec{\omega}(s)$ gives rise to zero potential throughout the region inside S_0 then from (17)

$$\vec{\rho}'(P)\vec{g} = \phi(P) = 0 \tag{38}$$

for this particular surface distribution. Now, when the potentials on the electrodes are given to be $v_1, v_2, v_3, \ldots v_m$, the arbitrary constants $x_0, x_1, x_2, \ldots x_m$, can be determined completely from the equation (29) but for an arbitrary constant α. From (16) the surface density $\vec{\sigma}(s)$ can be expressed in terms of the electrode

*As a non-trivial solution of its Transposed equation exists.

potentials $v_1, v_2, \ldots v_m$ but not uniquely because the arbitrary constant α remains undetermined from the boundary conditions, thus

$$\vec{\sigma}(s) = N(s)\vec{x} = N(s)P'U^{-1}\vec{v} + \alpha N(s)\vec{g} \qquad \ldots \quad (39)$$

From (17), the potential function $\phi(P)$ at the point P is

$$\begin{aligned} \phi(P) &= \vec{\rho}'(P)\vec{x} \\ &= \vec{\rho}'(P)P'U^{-1}\vec{v} + \alpha\vec{\rho}'(P)\vec{g} \\ &= \vec{\rho}'(P)\, P'U^{-1}\vec{v} \end{aligned} \qquad \ldots \quad (40)$$

the second term vanishes as has been shown just now. This is the complete solution of the boundary value problem and it is also unique The apparent paradox is thus removed. Though the surface distribution cannot be determined uniquely, the potential function can be determined uniquely from the boundary conditions.

Since $\vec{\rho}'(P)$ assumes the values $\vec{p}_1', \vec{p}_2', \ldots \vec{p}_m'$ when the point P lies on the electrodes, the scalar products $\overline{p_1}'\overline{g_1}, \overline{p_2}'\overline{g}, \ldots \overline{p_m}'\overline{g}$ should vanish according to (35). Or in other words

$$P\vec{g} = 0$$

which has been proved independently in Appendix III.

INFINITELY EXTENDED MEDIUM AND CORRESPONDENCE WITH THE MAXWELL'S COEFFICIENTS OF CAPACITY

When the semi-conducting medium extends to infinity, the problem becomes somewhat modified. The bounding surface S_0 no longer exists and therefore the boundary condition (4 2) is not material in this case, and has to¯be replaced by the usual regularity conditions for the Newtonian Potential at infinity. Accordingly, the equation of constraint (5), does not exist for this case. Hence the problem becomes identical with the Dirichlet's Exterior Problem for many bodies and its solution is well known (Maxwell) Thus it reduces to the electrostatic problem for many bodies discussed by Maxwell. The matrices P and Q both become regular square matrices of $m \times m$ order. Consequently,

$$\vec{i} = \kappa QP^{-1}\vec{v}, \qquad \vec{v} = \frac{1}{\kappa} PQ^{-1}\vec{i}$$

in this case. The matrix QP^{-1} however, still remains symmetric, because Green's

Reciprocal Theorem can still be applied. The elements of QP^{-1} are obviously the usual Maxwell's Coefficients of capacity (Jeans, 1923).

THE PHYSICAL INTERPRETATION OF THE SOLUTION AND CROSS-CURRENTS BETWEEN ELECTRODES

The currents flowing through the electrodes and the potentials of the electrodes are the only observable physical quantities. The natural question is how to determine the matrices P and Q or their functions from those observable quantities. First following Kirchhoff (1893) let us give the first electrode unit potential keeping the rest at zero potential and designate the corresponding currents through them as $c_{11}, c_{21}, c_{31}, \ldots c_{m1}$. These currents must satisfy the fundamental property that their sum is zero, that is,

$$c_{11}+c_{21}+c_{31}+\cdots+c_{m1}=0$$

or

$$c_{11}=-c_{21}-c_{31}-\cdots-c_{m1}$$

Physically it is obvious that no cross-currents (which have not yet been defined mathematically) can flow between the second and the third electrodes and so on, because they are at the same potential. Actually, the current which enters the first electrode leaves through the other electrodes. The current that enters through the second electrode is c_{21} and therefore the 'cross-current' between the first and the second electrode is c_{21} and so on. Let us call $c_{12}, c_{22}, c_{32}, \ldots c_{m2}$ the currents flowing through the electrodes when the second is kept at unit potential and the rest at zero potential and so on. Then from the fundamental relation (28),

$$\kappa Q'\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}=P'\begin{bmatrix}c_{11}\\c_{21}\\ \vdots\\c_{m1}\end{bmatrix},\quad \kappa Q'\begin{bmatrix}0\\0\\ \vdots\\0\end{bmatrix}=P'\begin{bmatrix}c_{12}\\c_{22}\\ \vdots\\c_{m2}\end{bmatrix}\ \text{etc.}$$

$$\kappa Q'\begin{bmatrix}1 & 0 & 0\ldots0\\ 0 & 1 & 0\,.\,0\\ \cdots & \cdots & \cdots\cdots\\ 0 & 0 & 0\ldots1\end{bmatrix}=P'\begin{bmatrix}c_{11} & c_{12} & \cdots c_{1m}\\ c_{21} & c_{22} & \cdots c_{2m}\\ \cdots & \cdots & \cdots\\ c_{m1} & c_{m2} & \cdots c_{mm}\end{bmatrix}$$

that is, where $\kappa Q'=P'C$ say, where $C=(c_{ij})$ ... (41)

Physically, the non-diagonal elements of this matrix represent the reverse cross-currents.

Again from (41) and (29) and Appendix II,

$$\kappa PQ' = PP'C \equiv UC$$

or

$$C = \kappa U^{-1}PQ' = \kappa(U^{-1}PQ')' = C' \qquad \dots (42)$$

and from (26′)

$$C\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \kappa U^{-1}PQ' \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = 0 \qquad \dots (43)$$

Thus C is a symmetric matrix. Physically this means that the cross-current that flows from the second electrode to the first when the first is at unit potential is equal to the cross-current that flows from the first to the second when the second is kept at unit potential and so on.

So far the concept of cross-currents is quite obvious and simple. The next question is what are the cross currents when the first, second ... electrodes have any potentials $v_1, v_2, v_3 \dots v_m$. Physically such cross currents must exist and shall have the property that the current which enters an electrode must branch out as cross-currents to other electrodes. Further they will have anti-symmetric property that is, cross-current from the first to the second electrode will be equal and opposite to that from the second to the first. Our task is to spot this cross-current matrix. Now from (31)

$$\vec{i} = \kappa U^{-1}PQ'\vec{v} = C\vec{v}$$

$$= C\begin{bmatrix} v_1 & 0 & 0 \dots 0 \\ 0 & v_2 & 0 \dots 0 \\ \dots & \dots & \dots \\ 0 & 0 & 0 \dots v_m \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$= CV\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \dots (44)$$

where V is the diagonal matrix

$$V_{m\times m} \equiv \begin{bmatrix} v_1 & 0 & 0 \dots 0 \\ 0 & v_2 & 0 \dots 0 \\ 0 & 0 & v_3 \dots 0 \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ 0 & 0 & 0 \dots v_m \end{bmatrix} \qquad \dots (45)$$

Let
$$J_{m \times m} \equiv CV - VC \qquad \ldots \quad (46)$$

This is a skew-symmetric matrix and

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = CV \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} - VC \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$= \bar{C}v \qquad \text{from (43)}$$

$$= \vec{i} \qquad \text{from (44)}$$

or
$$j_{12} + j_{13} + \cdots + j_{1m} = i_1, \quad j_{21} + j_{23} + \cdots + j_{2m} = i_2$$

That is, the matrix J has the requisite fundamental property that the sum of the cross currents branching out from an electrode is equal to the current that enters it. Further the matrix J reduces to the column vectors of C' when $v_1 = 1$, $v_2 = 0$, $v_3 = 0, \ldots v_m = 0$ etc., that is, they become identical with the cross-currents for the physically obvious fact mentioned at the beginning of this section. More explicitly, the cross-current matrix J is

$$\begin{matrix} 0 & c_{12}(v_2 - v_1) & c_{13}(v_3 - v_1) \ldots & c_{1m}(v_m - v_1) \\ c_{21}(v_1 - v_2) & 0 & c_{23}(v_3 - v_2) \ldots & c_{2m}(v_m - v_2) \\ \vdots & & & \\ c_{m1}(v_1 - v_m) & c_{m2}(v_2 - v_m) \ldots\ldots\ldots & \ldots\ldots\ldots & 0 \end{matrix} \qquad (47)$$

The skew symmetry is consistent with the physical property of cross-currents, viz. the cross-current from the first electrode to the second is the reverse of that flowing from the second to the first.

The Kirchhoff's Laws take the following from for continuous media :

$$J = CV - VC \qquad (48)$$

$$J \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \vec{i} \qquad \ldots \quad (59)$$

where C is the conductivity and V the diagonal matrix defined eariier.

Physically, the Ohmic heat generated by the hypothetical cross-currents will be

$$\tfrac{1}{2}\sum_{p=1}^{m}\sum_{q=1}^{m} i_{pq}\,(v_p - v_q) = \tfrac{1}{2}\sum_{p=1}^{m} v_p \sum_{q=1}^{m} i_{pq} - \tfrac{1}{2}\sum_{q=1}^{m} v_q \sum_{p=1}^{m} i_{pq}$$

$$= \tfrac{1}{2}\sum_{p=1}^{m} v_p \sum_{q=1}^{m} i_{pq} + \tfrac{1}{2}\sum_{q=1}^{m} v_q \sum_{p=1}^{m} i_{qp}$$

$$= \tfrac{1}{2}\sum_{p=1}^{m} v_p i_p + \tfrac{1}{2}\sum_{q=1}^{m} v_q i_q$$

$$= \sum_{p=1} v_p i_p$$

which is identical with the expression for the actual heat generated. Thus, physically the cross-currents can replace the electrode currents for all purposes

MATHEMATICAL INTERPRETATION OF THE SOLUTION AND EXISTENCE OF CROSS-CURRENTS

In the preceding section we demonstrated the existence of cross-currents from purely physical considerations The cross-current matrix did not come out automatically from the solution of the integral equations (11) We shall now prove, by suitable choice of the solution of the integral equations (14), the existence of cross-currents as also the matrix equation (48) satisfied by them.

Any linear combination of the $(m+1)$ linearly independent solutions $\vec{v_0}(s)$, $\vec{v_1}(s)$, ... $\vec{v_m}(s)$ of the simultaneous homogeneous integral equations (11) is a solution Since these $(m+1)$ solutions are all linearly independent exactly $(m+1)$ linearly independent solutions can be constructed through linear combinations of these solutions This new set of solutions can be taken as the fundamental set. We choose for the first solution $\vec{v_0}(s)$ the peculiar characteristic solution $\vec{\omega}(s)$ of the problem (34) Thus,

$$v_{00}(s) = \omega_0(s),\ v_{10}(s) = \omega_1(s) = 0 \qquad v_{m0}(s) = \omega_m(s) = 0$$

Through this choice of solution the 0-th columns of the both the $m\times(m+1)$ matrix P and Q become null vectors. Let us denote by Π the $m\times m$ square matrix which is formed from P leaving the 0-th column which has become a null vector. The rank of this $m\times m$ matrix Π is full as the rank of the $m\times(m+1)$ matrix P was proved to be m Let us form a new set of linearly independent solutions

$$\Lambda(s) \equiv \left(\vec{\omega}(s), \vec{\lambda_1}(s), \vec{\lambda_2}(s), \ldots \vec{\lambda_m}(s)\right)$$

by linear combinations of $\vec{\nu}_1(s), \vec{\nu}_2(s), \ldots \vec{\nu}_m(s)$

$$\underset{(m+1)\times(m+1)}{\wedge(s)} = \begin{pmatrix} \vec{\omega}(s).N(s)\Pi^{-1} \\ (m+1)\times m \end{pmatrix}$$

For this fundamental solution let us denote by the matrix $\underset{m\times m}{\Theta}$ the integral,

$$\Theta = \iint \wedge^0(Q)dS$$

By choice of this fundamental solution $\wedge(s)$ the matrix P defined in (21) becomes the identity matrix I_m with null vector as its 0-th column Thus the symmetry relation (27).

$$P'Q = Q'P$$

reduces in this case to

$$\Theta = \Theta'$$

and the relation (32) reduces simply to

$$\vec{x} = \begin{pmatrix} 0 \\ v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \vdots \\ 0 \end{pmatrix}$$

Thus the vector

$$\begin{pmatrix} x_1 \\ x_2 \\ \\ , \\ x_m \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \\ , \\ v \end{pmatrix}$$

The diagonal matrix dia $(x_1, x_2, \ldots x_m)$ is

$$\text{dia}\,(x_1, x_2, \ldots x_m) = \text{dia}\,(v_1, v_2, \ldots v_m) \equiv V$$

and from (41)

$$C = \kappa\Theta$$

By this transformation, the Q matrix becomes a square and symmetric matrix Θ. The current i_1 flowing through the first electrode is given by

$$\begin{aligned}
\kappa \iint_{S_1} \sigma_1(Q)dS = i_1 &= \kappa\{x_1 \iint_{S_1} \lambda_{11}(Q)dS + x_2 \iint_{S_1} \lambda_{12}(Q)dS + \ldots\} \\
&= \kappa\{\theta_{11}x_1 + \theta_{12}x_2 + \ldots + \theta_{1m}x_m\} \\
&= \kappa\{-(c\theta_{21} + \theta_{31} + \ldots \theta_{m1})x_1 + \theta_{21}x_2 + \theta_{31}x_3 + \ldots \theta_{m1}x_m\} \\
&= \kappa\{\theta_{21}(x_2 - x_1) + \theta_{31}(x_3 - x_1) + \ldots \theta_{m1}(x_m - x_1)\} \\
&= \kappa\{(x_2 - x_1) \iint_{S_2} \lambda_{21}(Q)dS + (x_3 - x_1) \iint_{S_3} \lambda_{31}(Q)dS + \ldots\}
\end{aligned}$$

Mathematically this relation means that the surface integral over the first electrode surface is contributed by a sum of surface integrals over all the electrodes surfaces excepting the first one. The surface integral $\kappa \iint_{S_1} \sigma_1(Q)dS$ represents the total current flowing from the first electrode which equals the sum of currents flowing through the other electrode surfaces. Thus,

$$\kappa(x_2 - x_1) \iint_{S_2} \lambda_{21}(Q)dS = \kappa(x_2 - x_1)\theta_{21}$$

represents the cross-current flowing from the first to the second electrode surface which has been denoted by j_{21}. The cross-current matrix is obviously,

$$J = \kappa\Theta\,\mathrm{dia}\,(x_1, x_2, \ldots x_m) - \kappa\,\mathrm{dia}\,(x_1, x_2, \ldots x_m)\Theta$$

$$= CV - VC$$

DISCUSSION OF THE MAIN RESULTS

Before attempting to deduce the usual Kirchhoff's Laws it will be profitable to recapitulate the important results already deduced.

(a) We have deduced in Section 4, that when the electrodes are kept at the same potential no currents will flow from them. This confirms to the physical fact.

(b) The algebraic sum of the currents flowing through all the electrodes is nil.

(c) *The algebraic sum of the currents flowing in and flowing out from an electrode is nil.* Equation (49).

(d) The cross-current flowing between any two electrodes is proportional to the difference of potentials between these electrodes. Equation (48).

These are summarised in the following formulae already deduced which correspond to Kirchhoff's Laws for a continuous medium :

$$J = CV - VC \qquad \ldots (48)$$

$$\vec{i} = J\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \qquad \ldots (49)$$

where $\qquad C' = C, \quad V = \mathrm{dia}\,(v_1, v_2, \ldots v_m) \qquad \ldots (50)$

and $\qquad (1, 1, \ldots 1)C = 0 \qquad \ldots (51)$

The coefficients of the conductivity matrix C are difficult to calculate mathematically from the geometry of the system. Experimentally each coefficient c_{lk} of the matrix can be determined by keeping the l-th electrode at unit potential and the remaining electrodes connected to ground through ammeters. The current flowing from the k-th electrode through the ammeter to the ground is the cross-current from the l-th electrode to the k-th electrode and is c_{lk}. In this way all the elements of C can be experimentally determined.

The boundary value problem can likewise be solved when the total currents flowing through the electrodes are specified instead of the potentials on the electrodes. It can also be solved when potentials are specified on some of the electrodes and the total currents are specified for the remaining electrodes. As it will make this paper too lengthy the solutions are not presented here.

For discussing the usual Kirchhoff's Laws it is necessary to consider a special case of the problem. It is quite possible that only a few of the m electrodes are kept 'live' that is connected to sources of constant potential, the remaining are kept floating. This will not affect the conductivity matrix C which is entirely governed by the geometry of the system. Since these *dead* electrodes cannot draw any current from any source, no current can flow into them. If we enumerate these dead electrodes from k onwards up to m, then

$$i_k = 0, \quad i_{k+1} = 0, \quad \ldots \quad i_m = 0$$

However, cross-currents will flow from them according to equation (48). The potential on these dead electrodes can be determined from the equation,

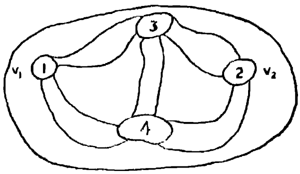
$$\begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_{k-1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = C \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{k-1} \\ \cdot \\ \cdot \\ \vdots \\ v_m \end{bmatrix}$$

in terms of the given $v_1, v_2 \ldots v_{k-1}$. If the currents $i_1, i_2, \ldots i_{k-1}$ are specified, the potentials can also be determined by partitioning the matrix C.

DEDUCTION OF THE USUAL KIRCHHOFF'S LAWS FOR LINE CONDUCTORS

Any net-work of resistors is actually some continuous media having some geometrical form. If we imagine the typical Wheatstones Bridge Network as a continuous medium, it will be somewhat like the following diagram :

There are four electrodes embeded in the semi-conducting medium having five lobes. The two electrodes (1) and (2) are connected to batteries and are at potentials v_1 and v_2, the two others (3) and (4) are 'dead' electrodes, but are at unknown potentials v_3 and v_4 to be determined. The cross-currents flowing between the electrodes (1) and (3) is $j_{13} = c_{13}(v_1 - v_3)$ Mathematically, c_{13} is difficult to determine from this geometry of the system. Since we are not interested in the micros-



copic distribution of currents in the material we take the macroscopic Ohm's Law. If R_{13} be the lumped resistance of the lobe, then macroscopic Ohm's Law states that the current between the electrodes (1) and (3) is $(v_1 - v_3)/R_{13}$ that is,

$$j_{13} = \frac{v_1 - v_3}{R_{13}}$$

From this relation we can at once get the value of the coefficient c_{13} of the conductivity matrix C for this case. Thus

$$c_{13} = \frac{1}{R_{13}}$$

which is identical with the total conductance of the lobe Similarly, we can find the other elements of the C matrix. If two electrodes are not connected through a lobe, then the corresponding coefficient will be zero, i e. $c_{12} = 0$ in this case, because the electodes (1) and (2) are not connected by any lobe of the matrial. Thus the conductivity matrix in this case is

$$\begin{bmatrix} -\left(\frac{1}{R_{13}} + \frac{1}{R_{14}}\right) & 0 & \frac{1}{R_{13}} & \frac{1}{R_{14}} \\ 0 & -\left(\frac{1}{R_{23}} + \frac{1}{R_{24}}\right) & \frac{1}{R_{23}} & \frac{1}{R_{24}} \\ \frac{1}{R_{13}} & \frac{1}{R_{23}} & -\left(\frac{1}{R_{13}} + \frac{1}{R_{23}} + \frac{1}{R_{24}}\right) & \frac{1}{R_{34}} \\ \frac{1}{R_{14}} & \frac{1}{R_{24}} & \frac{1}{R_{34}} & -\left(\frac{1}{R_{14}} + \frac{1}{R_{24}} + \frac{1}{R_{34}}\right) \end{bmatrix}$$

The number of independent current variables is $2-1 = 1$ in this case.

All the equations deduced for the continuous media remain the same. It is now apparent that equation (49) expresses Kirchhoff's Nodal Law. The usual Mesh Law follows as an obvious consequences of the matrix equation (48) if the conductances are replaced by resistances.

For alternating current circuits containing capacitors and inductances besides resistors, the usual Kirchhoff's Laws are extended by analogy. Through analogy the equations (48) and (49) can also be extended to the case of alternating currents. The only difference will be that the potentials and currents will be complex quantities and the elements of the conductivity matrix C will also be complex. This admittance matrix C will, however, remain symmetric and all other equations will remain the same. The complex elements of this admittance matrix can likewise be calculated from the branch impedances. This will be discussed in a future communication.

The number of independent variables are at once apparent from the generalised form of Kirchhoff's Laws. If there are n live electrodes, the number of independent voltage variables is n and the number of independent current variables is $n-1$.

APPENDIX 1

UNIQENESS OF SOLUTION

Suppose ψ be another solution besides ϕ

Let $$\chi = \phi - \psi$$

Thus, $$\nabla^2\chi = 0$$

and $\chi = 0$ on $S_1, S_2, \ldots S_m$ also

$$\left(\frac{\partial\chi}{\partial n}\right)_{S_0} = 0 \quad \text{on} \quad S_0$$

Now

$$\text{div}\,(\chi \text{ grad } \chi) = (\text{grad } \chi)^2 + \chi\nabla^2\chi(\text{grad } \chi)^2$$

By Green's Theorem,

$$\iiint (\text{grad } \chi^2)^2 d\tau = \iiint \text{div}(\chi \text{ grad } \chi) d\tau$$

$$= \sum_{i=0}^{m} \iint_{S_i} \chi \left(\frac{\partial\chi}{\partial n}\right)_{S_i} dS$$

$$= 0$$

Therefore

$$\text{grad } \chi = 0$$

Or,

$$\phi - \psi \equiv \chi = \text{constant} = 0$$

since

$$\chi = 0, \text{ on } S_1, S_2 \ldots S_m$$

Hence

$$\phi = \psi$$

Equation of constraint

Since $\nabla^2\phi = 0$ in the region formed by the surfaces $S_0, S_1, S_2, \ldots S_m$ then by Green's Theorem,

$$0 = \iiint \nabla^2\phi d\tau = \sum_{i=0}^{m} \iint_{S_i} \left(\frac{\partial\phi}{\partial n}\right)_{S_i} dS$$

$$= \sum_{i=1}^{m} \iint_{S_i} \left(\frac{\partial\phi_+}{\partial n}\right)_{S_i} dS + \iint_{S_0} \left(\frac{\partial\phi_-}{\partial n}\right)_{S_0} dS$$

since by the boundary condition (4.2) $\left(\frac{\partial\phi}{\partial n}\right)_{S_0} = 0$

$$\sum_{i=1}^{m} \iint_{S_i} \left(\frac{\partial\phi_\pm}{\partial n}\right)_{S_i} dS = 0$$

Every solution of the boundary value problem has to satisfy this equation of constraint. Physically this means that the algebraic sum of the total currents flowing through the electrode must vanish.

APPENDIX II

REGULARITY OF THE MATRIX

Since P' is a matrix of $(m+1)$ rows and m columns, the matrix $PP' \equiv U$ has m rows and m columns and is symmetric square matrix. Let $\overrightarrow{p'}_1, \overrightarrow{p'}_2, \ldots \overrightarrow{p}_m$ denote the m row vectors of P. Let us suppose that there is a linear dependence amongst them, viz.,

$$c_1\overrightarrow{p'}_1 + c_2\overrightarrow{p'}_2 + \ldots + c_m\overrightarrow{p'}_m = 0$$

But from the boundary conditions (22),

$$v_1 = p_{10}x_0 + p_{11}x_1 + \cdots + p_{1m}x_m$$

$$v_2 = p_{20}x_0 + p_{21}x_1 + \cdots + p_{2m}x_m$$

$$v_m = p_{m0}x_0 + p_{m1}x_1 + \cdots + p_{mm}x_m$$

Multiplying by $c_1, c_2, \ldots c_m$ respectively these equations and adding,

$$c_1v_1+c_2v_2+\ldots+c_mv_m = (c_1p_{10}+c_2p_{20}+\ldots+c_mp_{m0})x_0+(c_1p_{11}+c_2p_{21}+\ldots+c_mp_{m1})x_1$$

$$= 0 \text{ by the above assumption}$$

If the assumption is true then there must always be a definite linear relation between the potentials of the electrodes, which is absurd. This will mean that if we give a potential to one electrode keeping the remaining at zero potential the solution will not exist. The potentials on the electrodes, however can be arbitrarily given. This is contradictory to the Existence Theorem already proved. Thus there cannot be any linear dependence amongst the m row vectors of P. The rank of P is therefore full. Consequently, the rank of its Legendre Transform PP' is also full, that is PP' is a nonsingular matrix. Thus U^{-1} exists.

Symmetry of $U^{-1}PQ'$

Since from (27)

$$P'Q = Q'P$$

$$UQ = PP'Q = PQ'P$$

thus,

$$Q = U^{-1}PQ'P \text{ since } U^{-1} \text{ exists}$$

Or,

$$QP' = U^{-1}PQ'PP' = U^{-1}PQ'U$$

Thus,

$$QP'U^{-1} = U^{-1}PQ' = (QP'U^{-1})'$$

Again,

$$(U^{-1}PQ')' = QP'U^{-1} = U^{-1}PQ'$$

Therefore, the matrices $QP'U^{-1}$ and $U^{-1}PQ'$ are symmetric and identical. Moreover QP' and PQ' are connected by a similarity transformation.

APPENDIX III

As has been proved in Appendix II the rank of the $m\times(m+1)$ matrix P is m, and therefore the equation

$$\vec{v} = \overrightarrow{Px}$$

can always be solved for $x_0, x_1, x_2, \ldots x_m$, but with an arbitrary constant. Multiplying the above equation by P' we get

$$\overrightarrow{P'v} = P'\overrightarrow{Px}$$

Here $P'P$ is a $(m+1)\times(m+1)$ square matrix but of rank m. Thus one and only one non-trivial solution of the homogeneous equation,

$$P'\overrightarrow{Px} = 0$$

exists according to the Fundamental Theorem of Linear Algebra) Let us denote this non trivial solution by $\overline{g}$ which has $(m+1)$ elements. Now,

$$UP = PP'P$$

thus

$$U\overrightarrow{Pg} = P(P'\overrightarrow{P})g = 0$$

Since U^{-1} exists as shown in Appendix II, therefore,

$$\overrightarrow{Pg} = 0$$

Or in other words the vector $\overrightarrow{g}$ is orthogonal to the m row vectors of P. viz, $\overrightarrow{p_1'}, \overrightarrow{p_2'}, \ldots \overrightarrow{p_m'}$ which have $(m+1)$ elements each Obviously, the elements of $\overrightarrow{g}$ are proportional to the determinants,

$$\begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{vmatrix}, \quad - \begin{vmatrix} p_{10} & p_{12} & \cdots & p_{1m} \\ p_{20} & p_{22} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ p_{m0} & p_{m2} & \cdots & p_{mm} \end{vmatrix}, \ldots$$

because by the well-known theorem of determinants,

$$0 = \begin{vmatrix} p_{10} & p_{11} & p_{12} & \cdots & p_{1m} \\ p_{10} & p_{11} & p_{12} & \cdots & p_{1m} \\ p_{20} & p_{21} & p_{22} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{m0} & p_{m1} & p_{m2} & \cdots & p_{mm} \end{vmatrix} = p_{10}\times \begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{vmatrix} - p_{11}\times \begin{vmatrix} p_{10} & p_{12} & \cdots & p_{1m} \\ p_{20} & p_{22} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ p_{m0} & p_{m2} & \cdots & p_{mm} \end{vmatrix}$$

as the two rows are identical

Further,

$$U Q\vec{g} = (PP'Q)\vec{g}$$

$$= (PQ'P)\vec{g}$$

because from (27) $P'Q = Q'P$ and from the result just proved, viz., $P\vec{g} = 0$,

$$U Q\vec{g} = 0$$

Since, U^{-1} eixsts.

$$Q\vec{g} = 0$$

therefore the vector $\vec{g}$ is also orthogonal to the m row vectors of Q.

Again, since $P'P$ is a symmetric matrix, $\vec{g}$ is also a solution of the transposed equation that is

$$(P'P)'\vec{g} = P'P\vec{g} = 0$$

Further the scalar product,

$$\vec{g}'P'\vec{r} = (P\vec{g})'\vec{r} = 0$$

because $P\vec{g} = 0$ By the Fundamental Theorem of Linear Algebra the solution of the inhomogeneous equation

$$\overline{P'v} = P'P$$

always exists whatever $\vec{v}$ is

Since all the m rows of the rectangular matrix P are linearly independent, the right inverse of P exists (Zurmuhl, 1962) Let this be W, that is

$$PW = I_m$$

where I_m is the identity matrix of m elements. and W is a $(m+1) \times m$ matrix. Obviously,

$$W = P'U^{-1}$$

because, $$PW = PP'U^{-1} \equiv UU^{-1} = I_m$$

Thus the general solution of the inhomogeneous equation is,

$$\vec{x} = Wv+\vec{\alpha g}$$

$$= P'U^{-1}\vec{v}+\vec{\alpha g}$$

where α is any arbitrary constant which cannot be determined from the boundary conditions

REFERENCES

Courant and Hilbert, *Methoden der Mathematischen Physik*, Bd I, 2nd edition 1931, pp 5-6

Jeans, *Mathematical Theory of Electricity and Magnetism* 4th edition, 1923, p 312, 350-352

Kirchhoff, Gesammelte Abhandlungen pp 33, 151 (Also the articles by F. Auerbach in Winkelmann's *Handbuch der Physik* Vol 3, Part I, 1893 p 204. and article by P. Debye in the *Encyclopadie der mathematischen Wissenschaften*, Bd V. 2, p 395 et seq

Maxwell, *Treatise on Electricity & Magnetism* 3rd edition, Vol I, Chapter III.

Zurmuhl, Matrizen, 3rd edition 1962, pp 116-117.

Sternberg and Smith, *Theory of Potential and Spherical Harmonics*, 1952, Chap V, p 145, Chap XI.